

Bernoulli Problems

Ten Easy Steps

In a hydraulic system, moving oil has kinetic energy, which is proportional to the square of the velocity of the oil. The pump adds energy to the hydraulic fluid by raising its pressure. Gravity can also add energy if the hydraulic lines drop in elevation. Energy is lost through friction in pipes; flow through valves, orifices, and fittings; motors; and elevation increases. All of these energy losses can be measured as a drop in pressure. Jacob Bernoulli was a mathematician with little understanding of friction; he assumed that friction is negligible, and the energy in a fluid at one point of a hydraulic circuit equals the energy at a second point. If we include pumps, motors, and friction, we can modify Bernoulli's equation to say that the energy in a fluid at one point of a hydraulic system plus the energy added, minus the energy removed, equals the energy in a fluid at a second point.

The modified Bernoulli equation is derived in the textbook

$$\text{as } Z_1 + \frac{p_1}{\gamma} + \frac{v_1^2}{2g} + H_P - H_M - H_L = Z_2 + \frac{p_2}{\gamma} + \frac{v_2^2}{2g} .$$

Subscripts 1 and 2 refer to two different points in the hydraulic circuit.

We can use this equation to solve hydraulic circuit problems with a 10-step process.

Step 1 Draw the diagram & label pipe lengths, elevations, points of interest, directions of flow, etc.

Step 2 Write the Bernoulli equation & identify any terms that equal zero.

Step 3 Calculate fluid velocity from flow rate.

Step 4 Calculate Reynolds number, N_R . If this number is less than 2000, then we have laminar flow, and we can use the remaining equations. Turbulent flow requires a different solution for the friction factor.

Step 5 Calculate the friction factor, f .

Step 6 Calculate the equivalent length of the fittings & valves.

Step 7 Calculate head loss due to friction in the pipes, fittings, valves, and strainers: H_L .

Step 8 Calculate pump head and motor head, H_P and H_M (if applicable).

Step 9 Calculate pressure due to the weight of a fluid in a tank (if applicable).

Step 10 Assemble Bernoulli's equation from its parts, and solve.

Z = elevation change

p = pressure

γ = specific weight of the oil

v = velocity

g = acceleration of gravity

H_P = pump head

H_M = motor head

H_L = head loss due to friction in the lines

$$N_R = \frac{v D \rho}{\mu} = \frac{v D}{\nu}$$

v = velocity

D = inside diameter of the pipe

ρ = mass density

g = acceleration of gravity

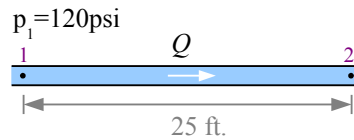
μ = absolute viscosity

ν = kinematic viscosity

Example #1

Oil flows at a rate of 7 gpm through a horizontal 1 inch ID pipe. The oil has a specific gravity $S.G. = 0.9$ and a kinematic viscosity $\nu = 100 \text{ cSt}$. If the pressure is 120 psi at one point, what is the pressure 25 feet downstream?

Step 1 Draw the circuit. There are no pumps, motors, fittings, valves, or elevation changes.



Step 2 Terms that go to zero in the Bernoulli equation include elevation change (because $Z_1 = Z_2$), velocity change (because $v_1 = v_2$), pump head (because there is no pump between points 1 and 2), and motor head (because there is no motor between points 1 and 2).

$$Z_1 + \frac{p_1}{\gamma} + \frac{v_1^2}{2g} + H_P - H_M - H_L = Z_2 + \frac{p_2}{\gamma} + \frac{v_2^2}{2g}$$

$$\frac{p_1}{\gamma} - H_L = \frac{p_2}{\gamma}$$

Step 3 Flow rate is volume per unit time: gallons per minute, or cubic meters per second. Velocity is distance per unit time: feet per minute, or meters per second. If you divide flow rate by cross-sectional area, you get velocity:

$$v = \frac{Q}{A} \text{ where } Q \text{ is flow rate.}$$

There is no leak of fluid between points 1 and 2, so the flow rate is the same at both points: $Q_1 = Q_2$. Since the pipe diameter is constant, the velocity is the same at both points: $v_1 = v_2$.

Step 4 The equation for Reynolds number depends on the units we're using for velocity, pipe diameter, and viscosity.

$$N_R = \frac{7740 v D S.G.}{\mu} \text{ for } v(\text{ft./s}), D(\text{in.}), \mu(\text{cP})$$

$$N_R = \frac{7740 v D}{\nu} \text{ for } v(\text{ft./s}), D(\text{in.}), \nu(\text{cSt})$$

$$N_R = \frac{1000 v D S.G.}{\mu} \text{ for } v(\text{m/s}), D(\text{mm}), \mu(\text{cP})$$

$$N_R = \frac{1000 v D}{\nu} \text{ for } v(\text{m/s}), D(\text{mm}), \nu(\text{cSt})$$

where 7740 and 1000 are constants. The textbook says "the constant contains the proper units to cause the Reynolds number to be dimensionless." The units of the constant

7740 are $\frac{\text{cSt s}}{\text{ft. in.}}$. The units of the constant 1000 are

$$\frac{\text{cSt s}}{\text{mm}}$$

Since $N_R < 2000$ we have laminar flow.

$$N_R = \frac{7740 v D}{\nu} = \frac{7740 \text{ cSt s}}{\text{ft. in.}} \frac{2.860 \text{ ft.}}{100 \text{ cSt}} \frac{1 \text{ in.}}{\text{s}} = 221.3$$

Step 5 Friction factor $f = \frac{64}{N_R}$. Like Reynolds number, the friction factor is unitless.

$$f = \frac{64}{221.3} = 0.289$$

Step 6 There are no fittings or valves, so the equivalent length of the system is the length of the pipe.

Step 7 The hydraulic fluid loses some energy due to friction as it passes through the pipe. Head loss

$H_L = f \frac{L}{D} \frac{v^2}{2g}$. The most common mistake at this step is forgetting to square the velocity.

Step 8 There is no pump or motor between points 1 and 2.

Step 9 There is no tank, so there is no additional pressure to calculate.

Step 10 Bernoulli's equation for this problem is

$\frac{p_1}{\gamma} - H_L = \frac{p_2}{\gamma}$ where γ is the specific weight of the oil, defined as $S.G._{oil} = \frac{\gamma_{oil}}{\gamma_{water}}$, so $\gamma_{oil} = S.G._{oil} \gamma_{water}$. Solving Bernoulli's equation for pressure at point 2,

$p_2 = \left[\frac{p_1}{\gamma} - H_L \right] \gamma$. The equation is easier to solve as $p_2 = p_1 - \gamma H_L$ because fewer unit conversions are needed.

The pressure change between the two points is $p_2 - p_1 = -4.3$ psi. The negative sign shows that the pressure has dropped from point 1 to point 2. In this problem, the pressure drop is due to friction in the pipe.

Example 2

Oil flows at a rate of 3 gpm through a horizontal 0.75 inch ID pipe for 10 feet, passes through a 90° standard elbow into a vertical pipe which drops for 12 feet, passes through a second 90° elbow into a horizontal pipe for another 14 feet. The oil has a specific weight $\gamma = 54$ lb./ft³ and a kinematic viscosity $\nu = 75$ cSt. If the initial pressure is 90 psi, what is the final pressure?

Step 1 Draw the circuit. There are no pumps, motors, or valves, but we have two fittings and an elevation change.

Step 2 Terms that go to zero in the Bernoulli equation include velocity change (because $v_1 = v_2$), pump head (because there is no pump between points 1 and 2), and motor head (because there is no motor between points 1 and 2).

Step 3 Velocity $v = \frac{Q}{A}$ where Q is flow rate.

There is no leak of fluid between points 1 and 2, so the flow rate is the same at both points: $Q_1 = Q_2$. Since the pipe diameter is constant, the velocity is the same at both points: $v_1 = v_2$.

Step 4 Since $N_R < 2000$ we have laminar flow.

$$L = L_{pipe} = 25 \text{ ft.}$$

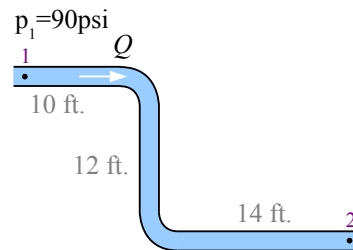
$$H_L = 0.289 \frac{25 \text{ ft.}}{1 \text{ in.}} \left| \frac{12 \text{ in.}}{\text{ft.}} \right| \frac{(2.860 \text{ ft./s})^2}{2(32.2 \text{ ft./s}^2)} = 11.01 \text{ ft.}$$

$$H_P = 0$$

$$H_M = 0$$

$$\gamma_{oil} = 0.9 \frac{62.4 \text{ lb.}}{\text{ft.}^3} = 56.16 \text{ lb./ft.}^3$$

$$p_2 = 120 \text{ psi} - \frac{56.16 \text{ lb.}}{\text{ft.}^3} \frac{11.01 \text{ ft.}}{\left| \frac{\text{ft.}^2}{(12 \text{ in.})^2} \right|} = 115.7 \text{ psi}$$



$$Z_1 + \frac{p_1}{\gamma} + \frac{v_1^2}{2g} + H_P - H_M - H_L = Z_2 + \frac{p_2}{\gamma} + \frac{v_2^2}{2g}$$

$$Z_1 + \frac{p_1}{\gamma} - H_L = Z_2 + \frac{p_2}{\gamma}$$

$$v_1 = \frac{3 \text{ gal.}}{\text{min.}} \left| \frac{4}{\pi(1 \text{ in.})^2} \right| \left| \frac{231 \text{ in.}^3}{\text{gal.}} \right| \left| \frac{\text{min.}}{60 \text{ s}} \right| \left| \frac{\text{ft.}}{12 \text{ in.}} \right| = 2.179 \text{ ft./s}$$

$$N_R = \frac{7740 \nu D}{v} = \frac{7740 \text{ cSt s}}{\text{ft. in.}} \frac{2.179 \text{ ft.} \cdot 0.75 \text{ in.}}{75 \text{ cSt s}} = 168.6$$

Step 5 Friction factor $f = \frac{64}{N_R}$.

Step 6 The equivalent length of the system is the length of the pipe plus the equivalent length of the two elbows.

Calculate the equivalent length of an elbow as $L_E = \frac{K D}{f}$

where D is the inside pipe diameter and K is the loss coefficient, or “ K factor” for the elbow. From the textbook, the loss coefficient of a 90° elbow is 0.75. We have two elbows, so KD/f is multiplied by 2.

Step 7 The hydraulic fluid loses some energy due to friction as it passes through the pipe and two elbows. Head loss $H_L = f \frac{L}{D} \frac{v^2}{2g}$.

Step 8 There is no pump or motor between points 1 and 2.

Step 9 There is no tank, so there is no additional pressure to calculate.

Step 10 Bernoulli’s equation for this problem is

$Z_1 + \frac{p_1}{\gamma} - H_L = Z_2 + \frac{p_2}{\gamma}$. Solving Bernoulli’s equation for

pressure at point 2, $p_2 = \left[(Z_1 - Z_2) + \frac{p_1}{\gamma} - H_L \right] \gamma$. In this equation, Z_1 and Z_2 are the elevations at the two points. Pick one of these elevations to be zero, then use the system diagram to determine the other elevation. For example, if point 1 has elevation $Z_1 = 0$, then point 2 has elevation $Z_2 = -12$ ft., and the change in elevation $Z_1 - Z_2 = 0$ ft. - (-12 ft.) = 12 ft.

What if we had picked point 2 as the zero elevation? Then point 1 would have an elevation of +12 feet, and the change in elevation $Z_1 - Z_2 = 12$ ft. - 0 ft. = 12 ft. ...the result is the same.

What if the flow direction were reversed? The math is the same, except $Z_1 - Z_2 = 0$ ft. - 12 ft. = -12 ft.

Now the pressure at point 2 is lower...there is a greater pressure drop because the oil is being pumped uphill.

$$f = \frac{64}{168.6} = 0.380$$

$$L = L_{pipe} + \frac{K D}{f}$$

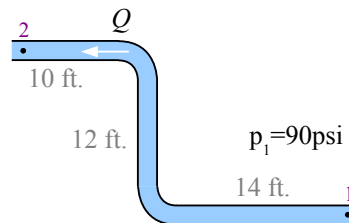
$$L = 10 \text{ ft.} + 12 \text{ ft.} + 14 \text{ ft.} + 2 \left[\frac{0.75 \cdot 0.75 \text{ in}}{0.380} \left| \frac{\text{ft.}}{12 \text{ in.}} \right. \right] = 36.25 \text{ ft.}$$

$$H_L = 0.380 \frac{36.25 \text{ ft.}}{0.75 \text{ in.}} \left| \frac{12 \text{ in.}}{\text{ft.}} \right. \frac{(2.179 \text{ ft./s})^2}{2(32.2 \text{ ft./s}^2)} = 16.22 \text{ ft.}$$

$$H_P = 0$$

$$H_M = 0$$

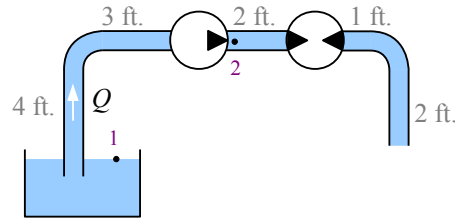
$$p_2 = \left[12 \text{ ft.} + \frac{90 \text{ lb.}}{\text{in.}^2} \frac{\text{ft.}^3}{54 \text{ lb.}} \left| \frac{(12 \text{ in.})^2}{\text{ft.}^2} - 16.22 \text{ ft.} \right. \right] \frac{54 \text{ lb.}}{\text{ft.}^3} \left| \frac{\text{ft.}^2}{(12 \text{ in.})^2} \right. = 88.4 \text{ psi}$$



$$p_2 = \left[-12 \text{ ft.} + \frac{90 \text{ lb.}}{\text{in.}^2} \frac{\text{ft.}^3}{54 \text{ lb.}} \left| \frac{(12 \text{ in.})^2}{\text{ft.}^2} - 16.22 \text{ ft.} \right. \right] \frac{54 \text{ lb.}}{\text{ft.}^3} \left| \frac{\text{ft.}^2}{(12 \text{ in.})^2} \right. = 79.4 \text{ psi}$$

Example 3

Oil flows at a rate of 8 gpm from a tank through 1 inch ID pipes and elbows, and through a pump and motor as shown. The pump adds 2 hp and the motor extracts 1 hp. The oil has a specific gravity $S.G. = 0.9$ and a kinematic viscosity $\nu = 98 \text{ cSt}$. What is the pressure at point 2, immediately downstream of the pump?



Step 1 Draw the circuit. There are no motors or valves, but we have a pump, one fitting and an elevation change.

Step 2 The velocity of the fluid at the surface of the tank is effectively zero, because its surface area is hundreds of times the cross-sectional area of the pipe, and because in a hydraulic system, all of the oil is ultimately returned to the tank, so the level remains constant.

The pressure at point 1, on the surface of the reservoir, is zero.

Since the motor is not between points 1 and 2, motor head is zero.

Step 3 Velocity $v_2 = \frac{Q}{A}$.

Step 4 Since $N_R < 2000$ we have laminar flow.

Step 5 Friction factor $f = \frac{64}{N_R}$.

Step 6 The equivalent length of the system is the length of the pipe plus the equivalent length of one elbow (the second elbow is not between points 1 and 2, so it is not included in the calculation).

Step 7 The hydraulic fluid loses some energy due to friction as it passes through the pipe and elbow. Head loss

$H_L = f \frac{L}{D} \frac{v^2}{2g}$. Since we'll need the term $\frac{v^2}{2g}$ later in Bernoulli's equation, let's calculate its value now.

Step 8 The pump adds 2 hp as it pressurizes the hydraulic fluid. From Chapter 3 we know that hydraulic power

$P = pQ$. Pressure is related to specific weight and head:
 $p = \gamma_{oil} H$, and $S.G._{oil} = \frac{\gamma_{oil}}{\gamma_{water}}$. Substituting, pump head

$H_P = \frac{p}{\gamma_{oil}} = \frac{P}{Q \gamma_{oil}} = \frac{P}{Q S.G._{oil} \gamma_{water}}$. If the power is in

horsepower and flow rate is in gpm, then

$H_P = \frac{3950 \text{ gpm ft.}}{\text{hp}} \frac{P}{Q S.G.}$. This equation also applies for motor head.

There is no motor between points 1 and 2, so motor head is zero.

$$Z_1 + \frac{p_1}{\gamma} + \frac{v_1^2}{2g} + H_P - H_M - H_L = Z_2 + \frac{p_2}{\gamma} + \frac{v_2^2}{2g}$$

$$Z_1 + H_P - H_L = Z_2 + \frac{p_2}{\gamma} + \frac{v_2^2}{2g}$$

$$v_2 = \frac{8 \text{ gal.}}{\text{min.}} \left| \frac{4}{\pi (1 \text{ in.})^2} \right| \left| \frac{231 \text{ in.}^3}{\text{gal.}} \right| \left| \frac{\text{min.}}{60 \text{ s}} \right| \left| \frac{\text{ft.}}{12 \text{ in.}} \right| = 3.268 \text{ ft./s}$$

$$N_R = \frac{7740 \nu D}{\nu} = \frac{7740 \text{ cSt s}}{\text{ft. in.}} \frac{3.268 \text{ ft.}}{98 \text{ cSt}} \frac{1 \text{ in.}}{\text{s}} = 258.1$$

$$f = \frac{64}{258.1} = 0.248$$

$$L = L_{pipe} + \frac{K D}{f}$$

$$L = 4 \text{ ft.} + 3 \text{ ft.} + \frac{0.75 \cdot 1 \text{ in.}}{0.248} \left| \frac{\text{ft.}}{12 \text{ in.}} \right| = 7.252 \text{ ft.}$$

$$\frac{v^2}{2g} = \frac{(3.286 \text{ ft./s})^2}{2(32.2 \text{ ft./s}^2)} = 0.1658 \text{ ft.}$$

$$H_L = 0.248 \frac{7.252 \text{ ft.}}{1 \text{ in.}} \left| \frac{12 \text{ in.}}{\text{ft.}} \right| \frac{0.1658 \text{ ft.}}{\text{ft.}} = 3.58 \text{ ft.}$$

$$H_P = \frac{3950 \text{ gpm ft.}}{\text{hp}} \frac{2 \text{ hp}}{8 \text{ gpm} \cdot 0.9} = 1097 \text{ ft.}$$

$$H_M = 0$$

Step 9 Neither point 1 nor point 2 lie at the bottom of the tank, so there is no need to calculate the pressure at the bottom of the tank.

Step 10 Bernoulli's equation for this problem is

$Z_1 + H_p - H_L = Z_2 + \frac{p_2}{\gamma} + \frac{v_2^2}{2g}$. Solving Bernoulli's

equation for pressure at point 2,

$$p_2 = \left[(Z_1 - Z_2) + H_p - H_L - \frac{v_2^2}{2g} \right] \gamma.$$

Example 4:

Oil flows at a rate of 12 gpm from a tank through 1 inch ID pipes and elbows, and through a pump and motor as shown. The strainer at the inlet has a pressure drop of 2 psi. The pump adds 3 hp and the motor extracts 1 hp. The oil has a specific gravity $S.G. = 0.9$ and a kinematic viscosity $\nu = 105 \text{ cSt}$. What is the pressure at point 2?

Step 1 Draw the circuit. We have a pump, a motor, two fittings, and an elevation change.

Step 2 The pressure and velocity of the fluid at the surface of the tank are zero, as in Example #3.

Step 3 Velocity $v_2 = \frac{Q}{A}$.

Step 4 Since $N_R < 2000$ we have laminar flow.

Step 5 Friction factor $f = \frac{64}{N_R}$.

Step 6 The equivalent length of the system is the length of the pipe plus the equivalent length of both elbows.

Step 7 The hydraulic fluid loses some energy due to friction as it passes through the strainer, pipe, and two elbows. The head loss due to friction in the pipes and elbows is $H_L = f \frac{L}{D} \frac{v^2}{2g}$. Since we'll need the term $\frac{v^2}{2g}$ later in Bernoulli's equation, let's calculate its value now.

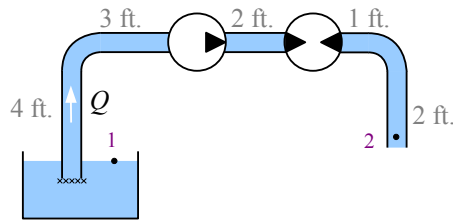
The head loss due to the strainer is $\frac{\Delta p}{\gamma}$. Therefore, the

total head loss is $H_L = f \frac{L}{D} \frac{v^2}{2g} + \frac{\Delta p}{\gamma}$.

$$Z_1 - Z_2 = 0 \text{ ft.} - 4 \text{ ft.} = -4 \text{ ft.}$$

$$p_2 = \left[-4 \text{ ft.} + 1097 \text{ ft.} - 3.58 \text{ ft.} - 0.166 \text{ ft.} \right] \frac{0.9 \cdot 62.4 \text{ lb.}}{\text{ft.}^3} \left| \frac{\text{ft.}^2}{(12 \text{ in.})^2} \right|$$

$$= 425 \text{ psi}$$



$$Z_1 + \frac{p_1}{\gamma} + \frac{v_1^2}{2g} + H_p - H_M - H_L = Z_2 + \frac{p_2}{\gamma} + \frac{v_2^2}{2g}$$

$$Z_1 + H_p - H_M - H_L = Z_2 + \frac{p_2}{\gamma} + \frac{v_2^2}{2g}$$

$$v_2 = \frac{12 \text{ gal.}}{\text{min.}} \left| \frac{4}{\pi (1 \text{ in.})^2} \right| \frac{231 \text{ in.}^3}{\text{gal.}} \left| \frac{\text{min.}}{60 \text{ s}} \right| \frac{\text{ft.}}{12 \text{ in.}} = 4.902 \text{ ft./s}$$

$$N_R = \frac{7740 \nu D}{\nu} = \frac{7740 \text{ cSt s}}{\text{ft. in.}} \frac{4.902 \text{ ft.}}{105 \text{ cSt}} \frac{1 \text{ in.}}{\text{s}} = 361.3$$

$$f = \frac{64}{361.3} = 0.177$$

$$L = L_{\text{pipe}} + \frac{K D}{f}$$

$$L = 4 \text{ ft.} + 3 \text{ ft.} + 2 \text{ ft.} + 1 \text{ ft.} + 2 \text{ ft.} + 2 \left[\frac{0.75 \cdot 0.75 \text{ in.}}{0.177} \left| \frac{\text{ft.}}{12 \text{ in.}} \right| \right]$$

$$= 12.71 \text{ ft.}$$

$$\frac{v^2}{2g} = \frac{(4.902 \text{ ft./s})^2}{2(32.2 \text{ ft./s}^2)} = 0.373 \text{ ft.}$$

$$H_L = 0.177 \frac{12.71 \text{ ft.}}{1 \text{ in.}} \left| \frac{12 \text{ in.}}{\text{ft.}} \right| \frac{0.373 \text{ ft.}}{\text{ft.}}$$

$$+ \frac{2 \text{ lb.}}{\text{in.}^2} \frac{\text{ft.}^3}{0.9 \cdot 62.4 \text{ lb.}} \left| \frac{(12 \text{ in.})^2}{\text{ft.}^2} \right| = 15.20 \text{ ft.}$$

Step 8 The pump adds 3 hp as it pressurizes the hydraulic fluid. From Example 3, $H_P = \frac{3950 \text{ gpm ft.}}{\text{hp}} \frac{P_{\text{pump}}}{Q \text{ S.G.}}$ and

$$H_M = \frac{3950 \text{ gpm ft.}}{\text{hp}} \frac{P_{\text{motor}}}{Q \text{ S.G.}}$$

Step 9 Neither point 1 nor point 2 lie at the bottom of the tank, so there is no need to calculate the pressure at the bottom of the tank.

Step 10 Bernoulli's equation for this problem is

$Z_1 + H_P - H_M - H_L = Z_2 + \frac{p_2}{\gamma} + \frac{v_2^2}{2g}$. Solving Bernoulli's equation for pressure at point 2,

$$p_2 = \left[(Z_1 - Z_2) + H_P - H_M - H_L - \frac{v_2^2}{2g} \right] \gamma$$

Bernoulli's equation shows us where the energy is added to the system and where it is used or lost. In this problem, the pump adds 1097 ft. of head; all losses total 383 ft. of head. Elevation change consumes 0.5% of the 383 ft., the motor consumes 95.4%, friction consumes 4.0%, and the remaining 0.1% is used to move the fluid (kinetic energy).

$$H_P = \frac{3950 \text{ gpm ft.}}{\text{hp}} \frac{3 \text{ hp}}{12 \text{ gpm} \cdot 0.9} = 1097 \text{ ft.}$$

$$H_M = \frac{3950 \text{ gpm ft.}}{\text{hp}} \frac{1 \text{ hp}}{12 \text{ gpm} \cdot 0.9} = 366 \text{ ft.}$$

$$Z_1 - Z_2 = 0 \text{ ft.} - 2 \text{ ft.} = -2 \text{ ft.}$$

$$p_2 = [-2 \text{ ft.} + 1097 \text{ ft.} - 366 \text{ ft.} - 15.2 \text{ ft.} - 0.373 \text{ ft.}] \\ \times \frac{0.9 \cdot 62.4 \text{ lb.}}{\text{ft.}^3} \left| \frac{\text{ft.}^2}{(12 \text{ in.})^2} \right| = 278 \text{ psi}$$